3D Reconstruction Algorithms
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Tomography
historical background

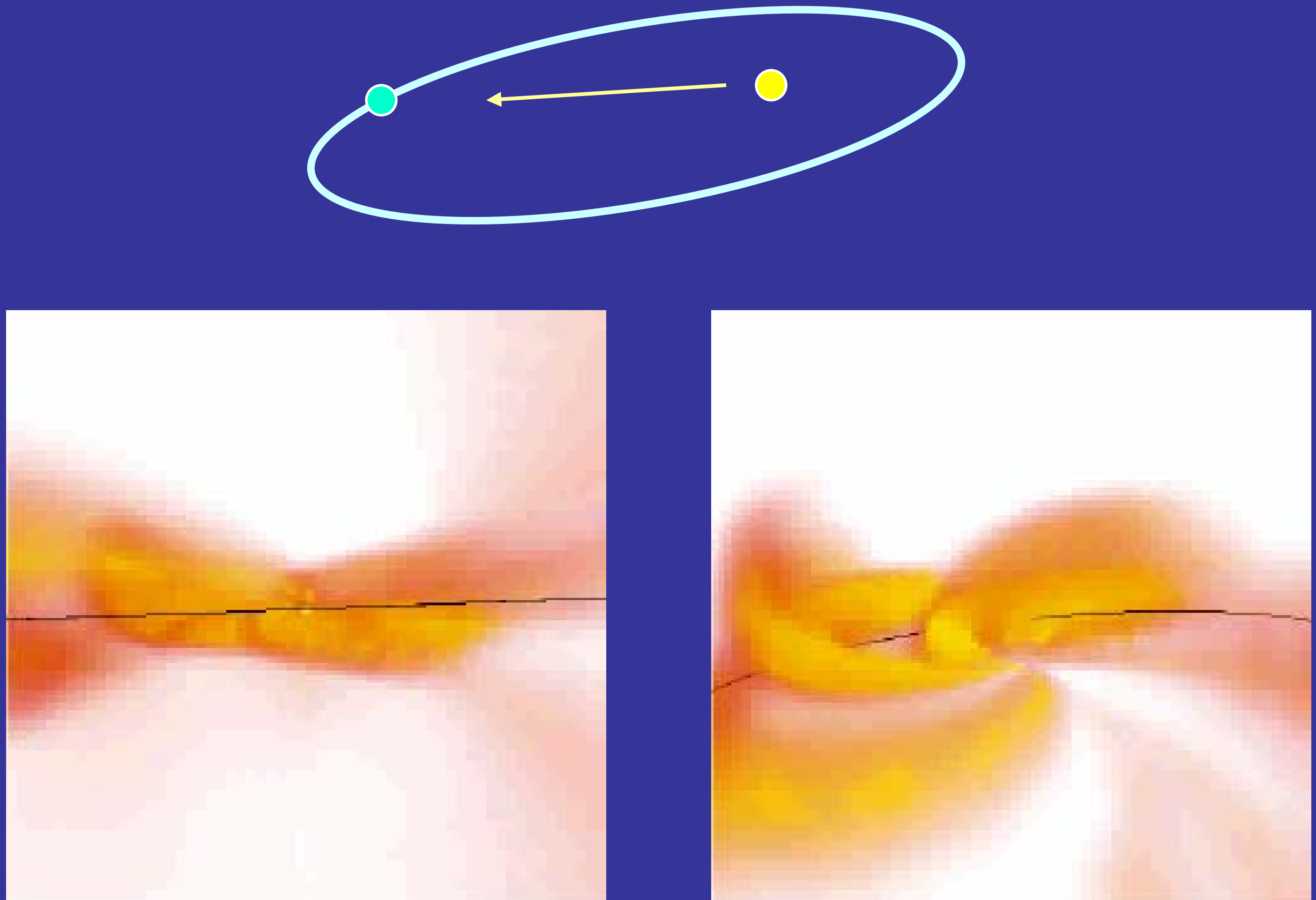
- 1956 - Bracewell reconstructed sun spots from multiple views of the Sun from the Earth.

- 1967 - Medical Research Council Laboratory, Cambridge, England: Aaron Klug and grad student David DeRosier reconstructed three-dimensional structures of viruses.

- 1972 - British engineer Godfrey Hounsfield of EMI Laboratories, England, and independently South African born physicist Allan Cormack of Tufts University, Massachusetts, invented CAT (Computed Axial Tomography) scanner. Tomography is from the Greek word *tomos* meaning "slice" or "section" and *graphia* meaning "describing".

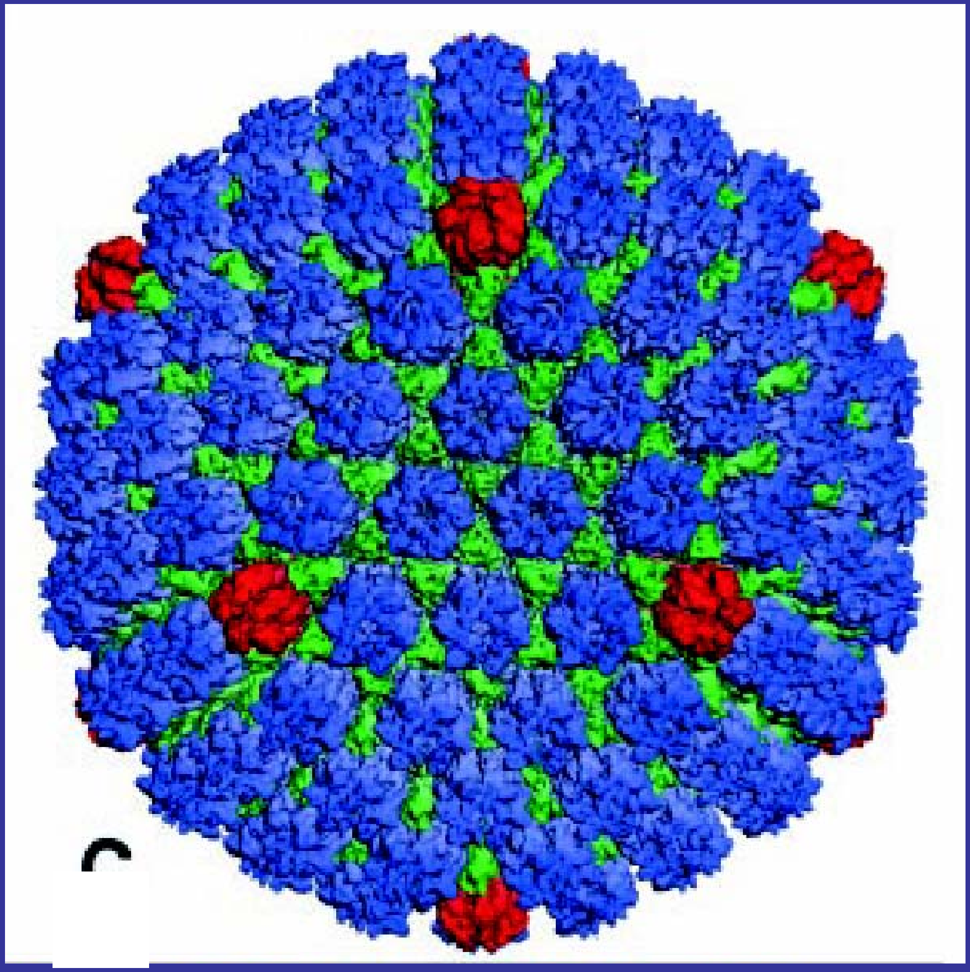
- 1977 – W. Hoppe (Germany) proposed three-dimensional high resolution electron microscopy of non-periodic biological structures (single particle analysis).

Inner heliospheric plasma density (to 1.5 times the distance of the Earth from the Sun).



Herpesvirus at 8.5 Å resolution

Zhou, Z. H., Dougherty, M., Jakana, J., He, J., Rixon, F. J. and Chiu, W. (2000) Seeing the herpesvirus capsid at 8.5 Å. *Science* **288**, 877-80.



Computed Axial Tomography
CAT Scan
or simply CT Scan

Just as the computer has had a major impact on the world of business, so has it had a major impact in Radiology. The CT Scan is one of those by-products. CT stands for Computerized Tomography and can best be defined as computers acquiring images of sections or "slices" of specific areas of your body. Your examination will produce detailed studies by stacking the individual "slices", creating an image of your anatomy.

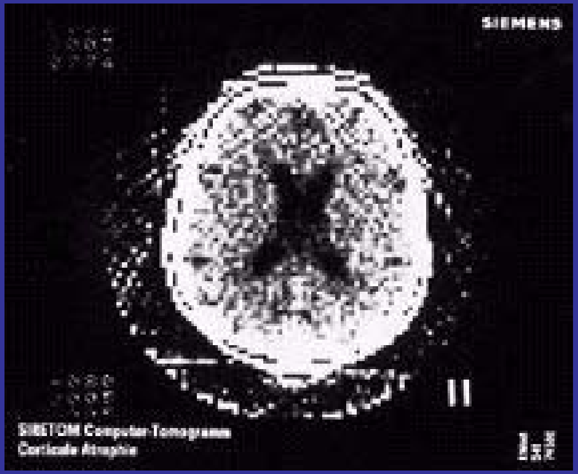
CT scans are used for many diagnostic procedures. For example, conventional X-rays cannot show brain structures. A CT scan can detect disorders in the brain with a great deal of accuracy. In scans of the body, a CT can determine which is bone, tissue, fat, gas and fluids. CT can determine if something is a solid or liquid, and if an organ's size and shape is normal.

CT Scan - X-ray technique that allows relatively safe, painless, and rapid diagnosis in previously inaccessible areas of the body; also called CT scan. An X-ray tube, rotating around a specific area of the body, delivers an appropriate amount of X radiation for the tissue being studied and takes pictures of that part of the internal anatomy from different angles. More recent scanners have a stationary X-ray tube and use deflecting coils and special reflectors to position the X-ray beam. A computer program is then used to form a composite, readable image. CAT scanning has revolutionized medicine, especially neurology, by facilitating the diagnosis of brain and spinal cord disorders, cancer, and other conditions. Ultrafast CT, or electron beam CT, is able to take pictures in a tenth of a second. It is useful in creating images of moving parts, such as the heart, without blurring.

CT scan

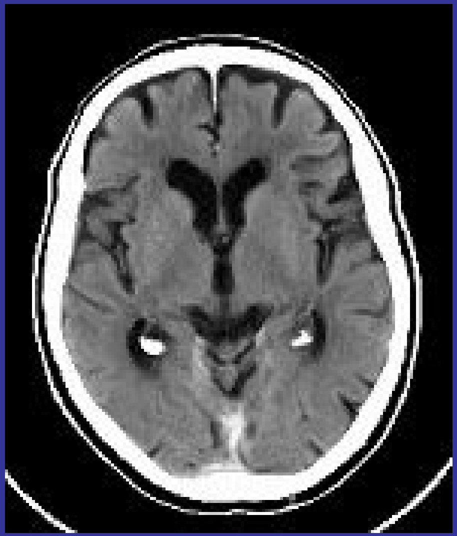
Original "Siretom" dedicated head CT scanner, circa 1974.

The first clinical CT scanners were installed between 1974 and 1976. The original systems were dedicated to head imaging only, but "whole body“ systems with larger patient openings became available in 1976. CT became widely available by about 1980. There are now about 6,000 CT scanners installed in the U.S. and about 30,000 installed worldwide.



Original axial CT image from the dedicated Siretom CT scanner, circa 1975.

This image is a coarse 128 x 128 matrix; however, in 1975 physicians were fascinated by the ability to see the soft tissue structures of the brain, including the black ventricles for the first time (enlarged in this patient) (Courtesy: Siemens)

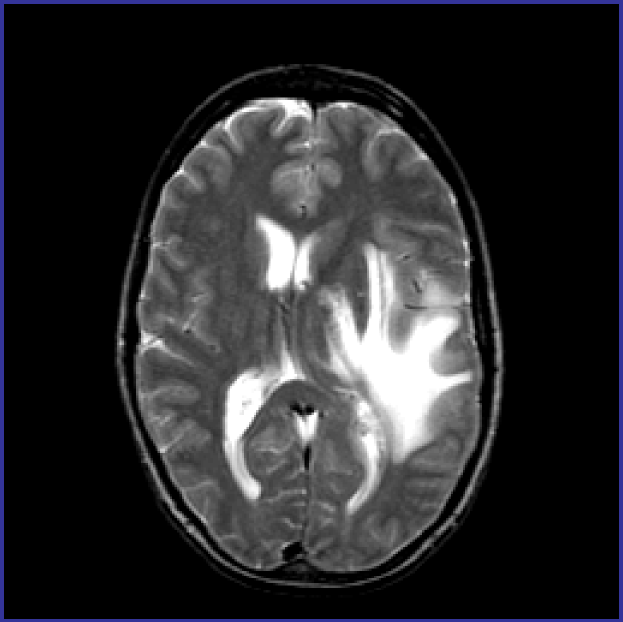


Axial CT image of a normal brain using a state-of-the-art CT system and a 512 x 512 matrix image.

Note the two black pea-shaped" ventricles in the middle of the brain and the subtle delineation of gray and white matter. (Courtesy: Siemens)

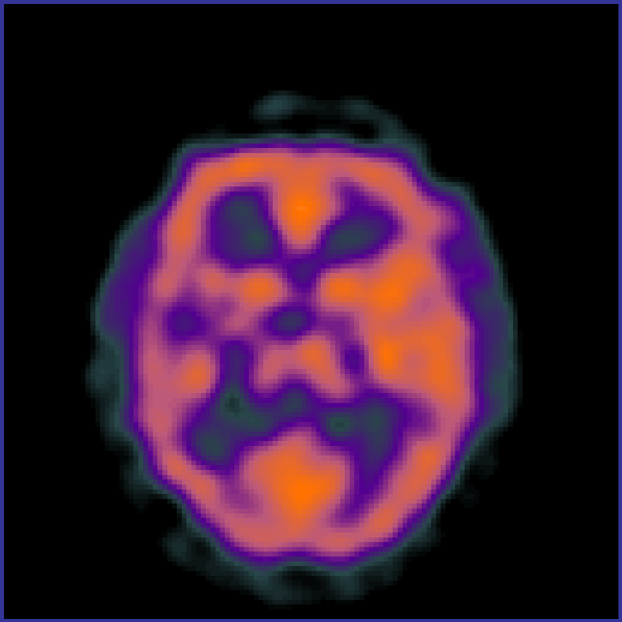
Various physical effects can be used to visualize different aspects of the human body physiology

X-rays



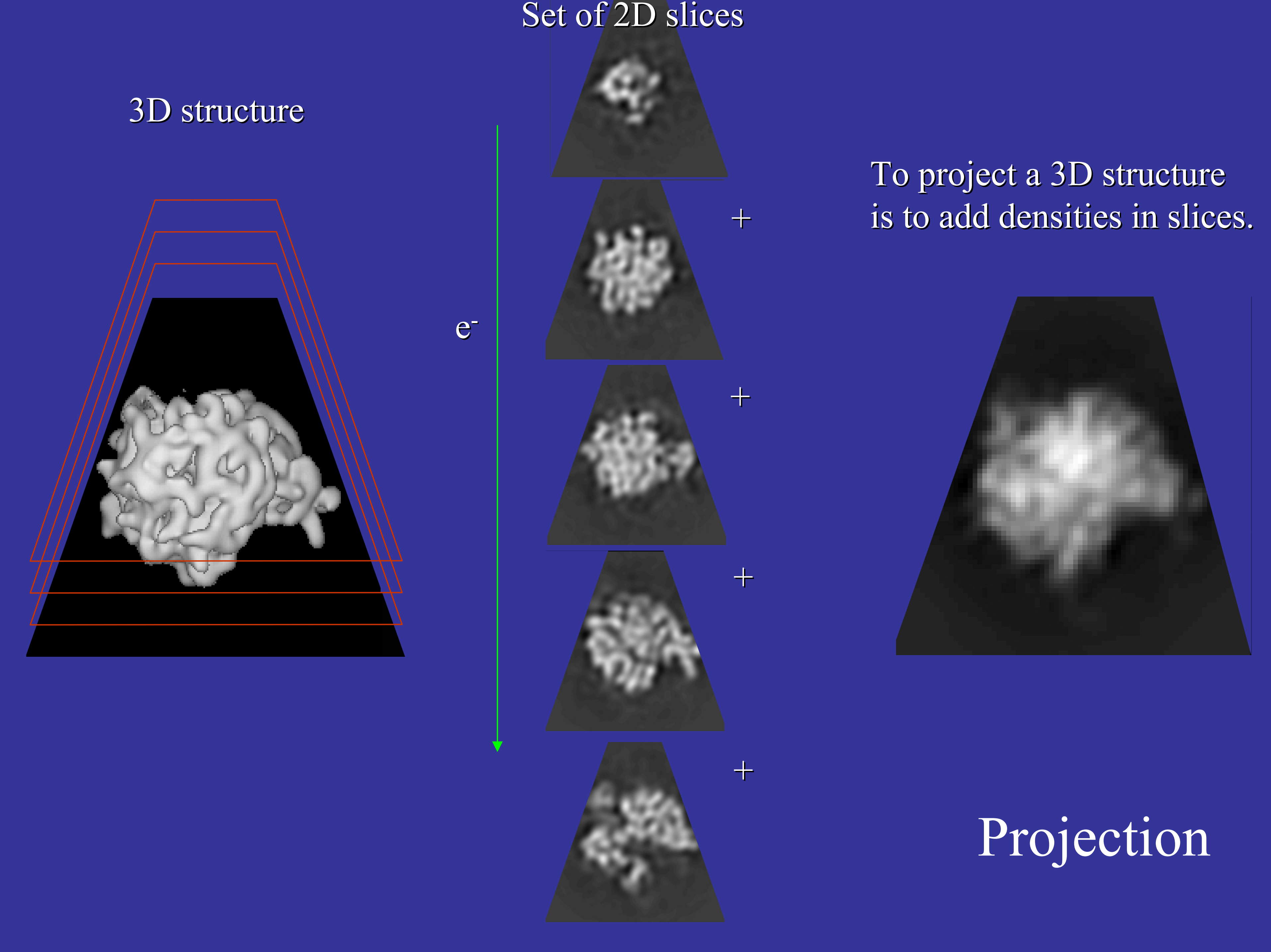
NMR

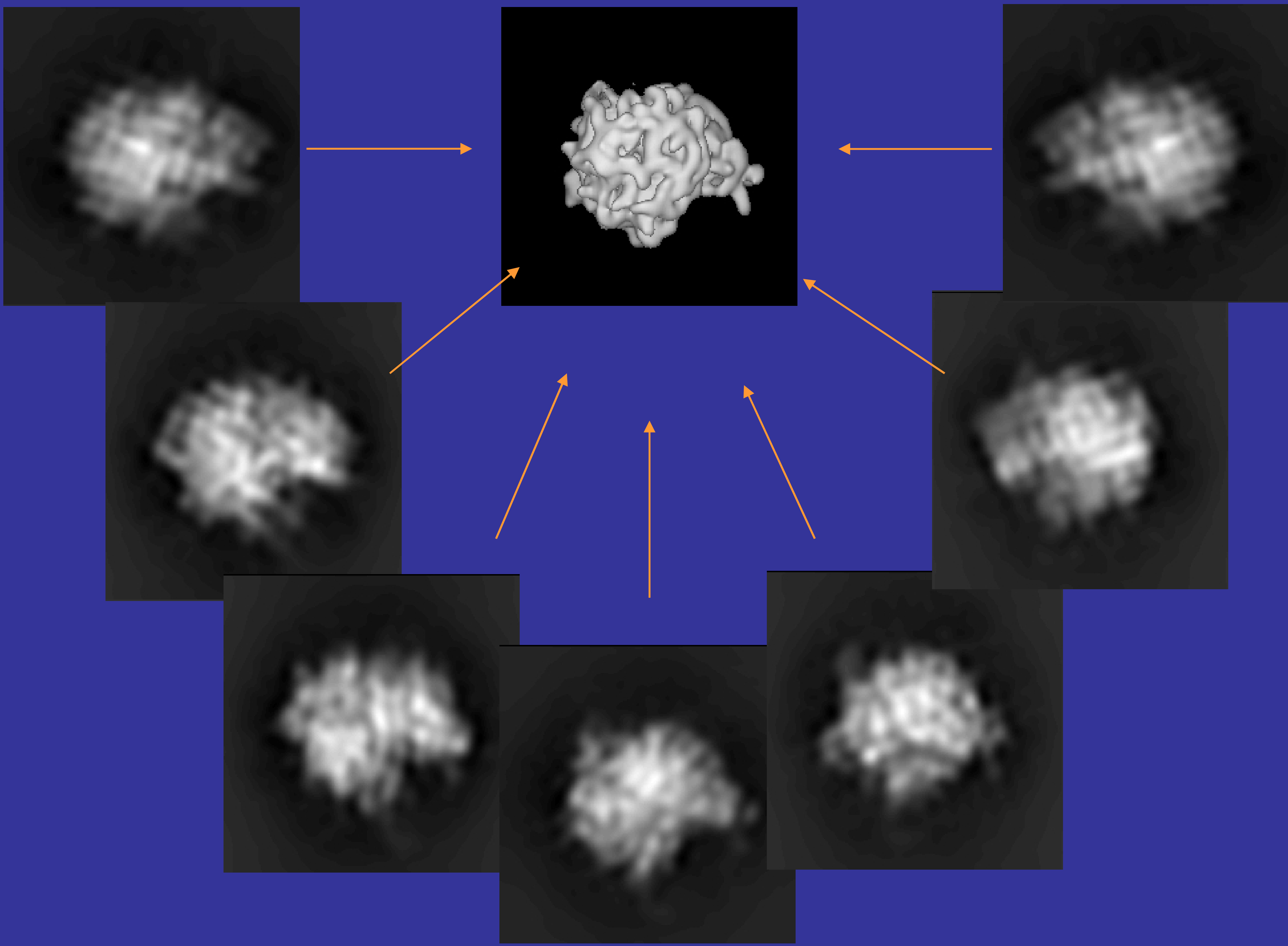
Nuclear Magnetic Resonance



PET

Positron Emission Tomography

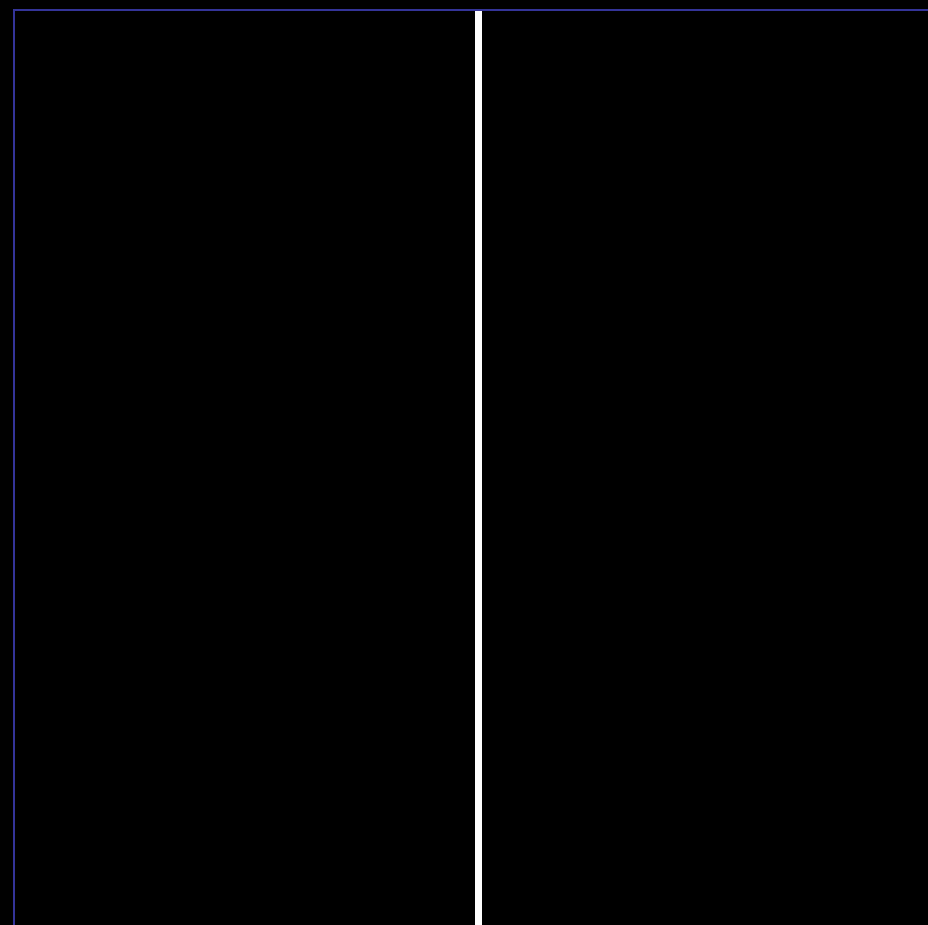

Set of 2D slices
3D structure
To project a 3D structure
is to add densities in slices.
e-
+
+
+
+
Projection

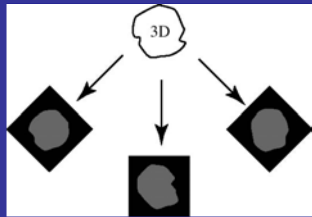


3D reconstruction(Back Projection)

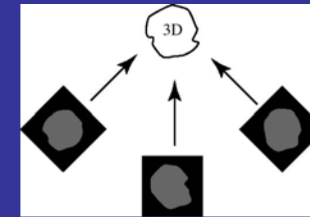
Full range

The process of tomographic reconstruction





Mechanism of projection-backprojection



0
1
0

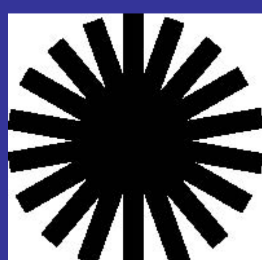
0	0	0
0	1	0
0	0	0

0	1	0

0
1
0

0	1	0
1	2	1
0	1	0

0	1	0

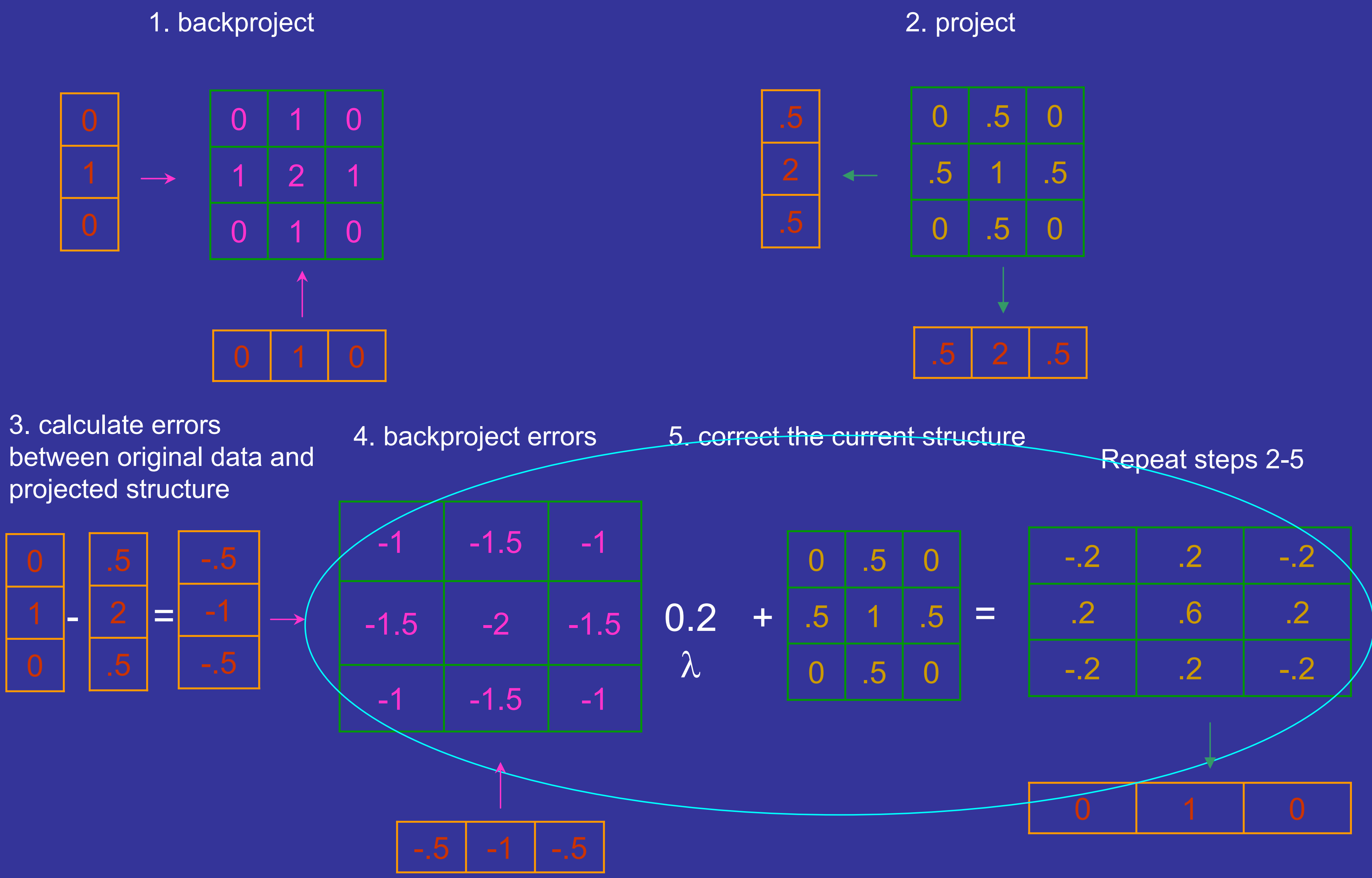


-1
1
-1

-2	0	-2
0	2	0
-2	0	-2

-1	1	-1

Iterative improvement of the reconstruction



3D reconstruction algorithm can be considered the most important element of the single particle reconstruction process

Many steps of the process are best understood in terms of the 3D reconstruction problem:

- construction of an initial model
- refinement of the structure
- resolution estimation

The problem of 3D reconstruction from projections in EM is substantially different from the problem of "classical" tomography:

- data collection geometry cannot be controlled (random distribution of projection directions)
- extremely uneven distribution of projection directions, in many cases resulting in gaps in Fourier space
- extremely low SNR
- large errors in orientation parameters, both random and systematic, in principle the 3D reconstruction should be a part of orientation refinement procedure
- number of projection data much larger than the linear size of projections

Why the problem of 3D reconstruction from projections remains interesting?

- ✓ The problem is ill-posed – small change in the input data (2D projections) can cause large change in the results (3D structure).
- ✓ Unique solution does not exist!
- ✓ Various experimental situation may require different 3D reconstruction algorithms depending on the required accuracy of the results, amount of the input data, time constraints….

Ghosts do exist

(a theorem)

For any set of projection directions there exists a non-trivial object $f_0 \neq 0$ such that its projections at given directions are zero, that is $Pf_0 = 0$.

X=a+b	a -1	b+1
Y=c+d	c +1	d -1
	U=a+c	V=b+d

a ghost

-1	1
1	-1

If ghosts exist, why we do not see them?

How to evaluate the quality of 3D reconstruction methods?

Figure of merit (FOM): a quantifiable similitude of the phantom (test object) with the reconstruction volume.

For a discussion and examples see:
Matej *et al.,* IEEE Trans. Image Process. **5** (1996) 554.
Marabini *et al.,* Ultramicroscopy **72** (1998) 53.

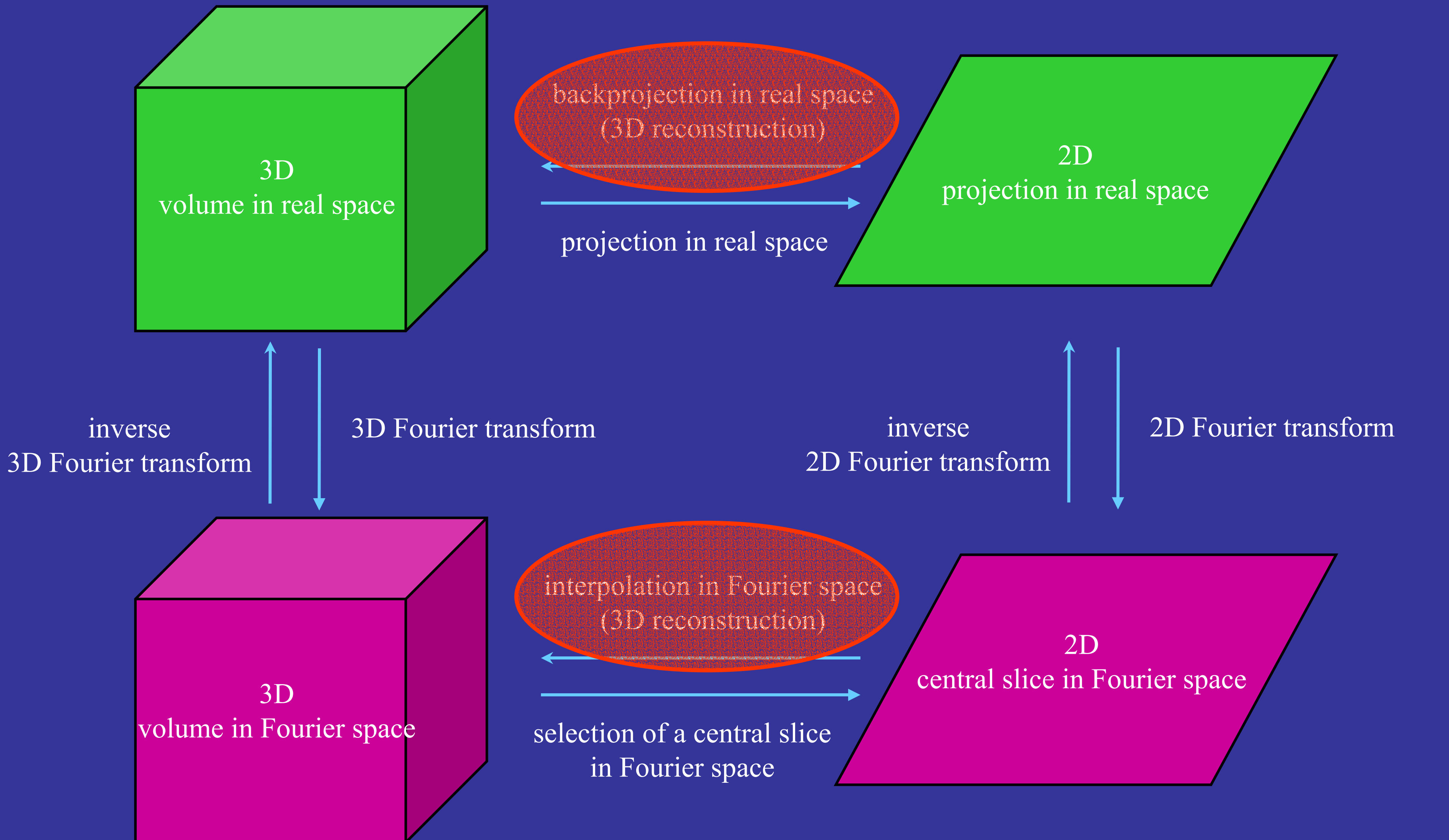
Possible FOM: **correlation coefficient**.

Measure of quality (FOM) of a 3D reconstruction method should be the same as the measure used to evaluate quality of the single-particle reconstruction process: the ***resolution*** measure.

Fourier Shell Correlation (FSC).

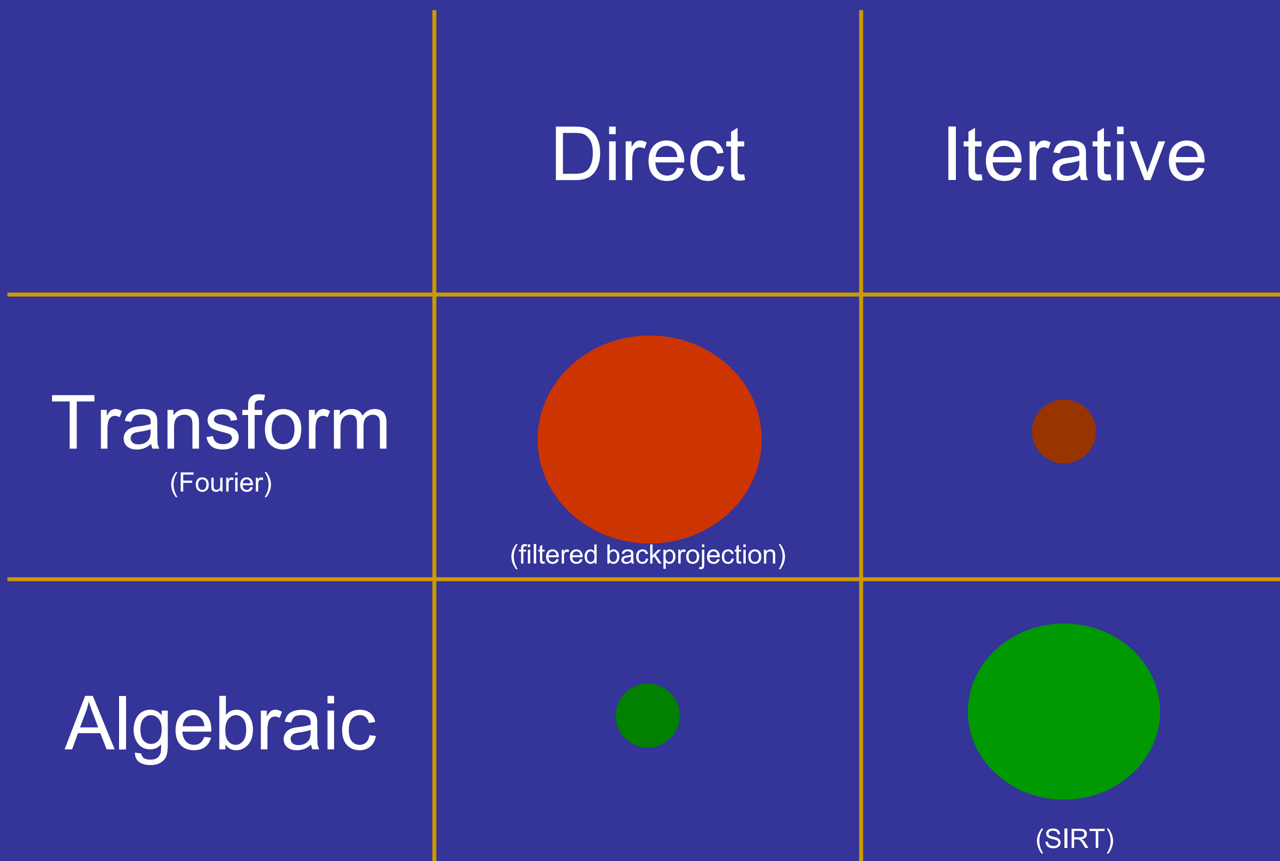
Twiddle knobs.

Tomography (reconstruction from projections).



Difficult *inverse problems*, exact inversion does not exist!

Taxonomy of tomographic methods



Algebraic methods

$$\text{Find vector } \tilde{f} \text{ that minimizes } L(f) = \|Pf - g\|^2.$$

Find such a 3D structure which 2D projections are most similar (in the Least Squares sense) to given 2D data.

Algorithms:

ART – Algebraic Reconstruction Technique.
Karczmarz's row action iterative algorithm for solving a system of linear equations.

SIRT – Simultaneous Iterative Reconstruction Technique:
(1) chose initial 3D structure $f^{(0)}$ (usually zero);
(2) modify 3D structure by a gradient $\nabla L(f)$
(3) repeat step 2 until convergence is reached.

Algebraic methods
SIRT

$$f^{(k+1)} = f^{(k)} - \lambda^{(k)} \nabla L(f^{(k)}) = f^{(k)} - \lambda^{(k)} \underset{2D \Rightarrow 3D}{P^T} \left(\underset{\underset{3D \Rightarrow 2D}{\cup}}{Pf^{(k)}} - \underset{2D}{g} \right)$$

$$f^{(k+1)} = f^{(k)} - \lambda^{(k)} \left(\underset{3D \Rightarrow 2D \Rightarrow 3D}{P^T P f^{(k)}} - \underset{3D}{P^T g} \right)$$

Depending on the way sequence $\left\{\lambda^{(k)}\right\}$ is chosen, we have three different algorithms:

$\lambda^{(k)} = \lambda = const$ - Richardson's method

$\lambda^{(k)} = \arg\min_{\lambda \geq 0} L\left(f^{(k)} - \lambda \nabla L\left(f^{(k)}\right)\right)$ - steepest descent

- conjugate gradient (requires construction of Q-conjugate directions).

Important features:

For the SIRT algorithm, the solution does not depend on the starting point.

Rate of convergence: Richardson's - 100; steepest descent - 25; conjugate gradient - 10.

Conjugate gradient method requires regularization (additional term, additional parameter).

Regularization for iterative algorithms: terminate iterations before convergence is reached.

Twiddle knobs:

Richardson's – number of iterations and λ.
Steepest descent – number of iterations.
Conjugate gradient – weight of regularizing term.

The parameters have to be adjusted for each data set separately.

If SIRT algorithms are slow and inconvenient, why would we want to use them?

- The quality of results surpasses the quality of results of other methods, particularly of those based on Fourier transform.

- SIRT algorithms perform better in “extreme” situations, such as uneven distribution of projections, incomplete projections (“missing cone”, “missing wedge”), reconstruction from few directions.

- SIRT algorithms are flexible. It is possible to incorporate additional constraints (positivity, limited spatial support), *a priori* knowledge, CTF correction….

Direct Fourier algorithms

- 1) **Back-Projection of Filtered Projections**
  - \- for each 2D projection construct a 2D filter taking into account distribution of remaining projections
  - \- filter each 2D projection using respective 2D filters
  - \- back-project filtered 2D projections (in real space)
-
- 2) **Filter of the Back-Projection**
  - \- back-project original 2D projections (in real space)
  - \- construct a 3D filter taking into account distribution of projections
  - \- apply the 3D filter to the 3D volume obtained from back-projected, unfiltered projections
- 3) **Interpolation in Fourier space**
  - \- calculate 2D Fourier transform and using a chosen interpolation scheme place it within a 3D (Fourier) volume

    \- calculate inverse 3D Fourier transform

Important features:

Fast.

Quality largely varies depending on the method and implementation.

Twiddle factors:
Usually hidden from the user. For a given parameter value, algorithms perform equally well in a broad range of situations.

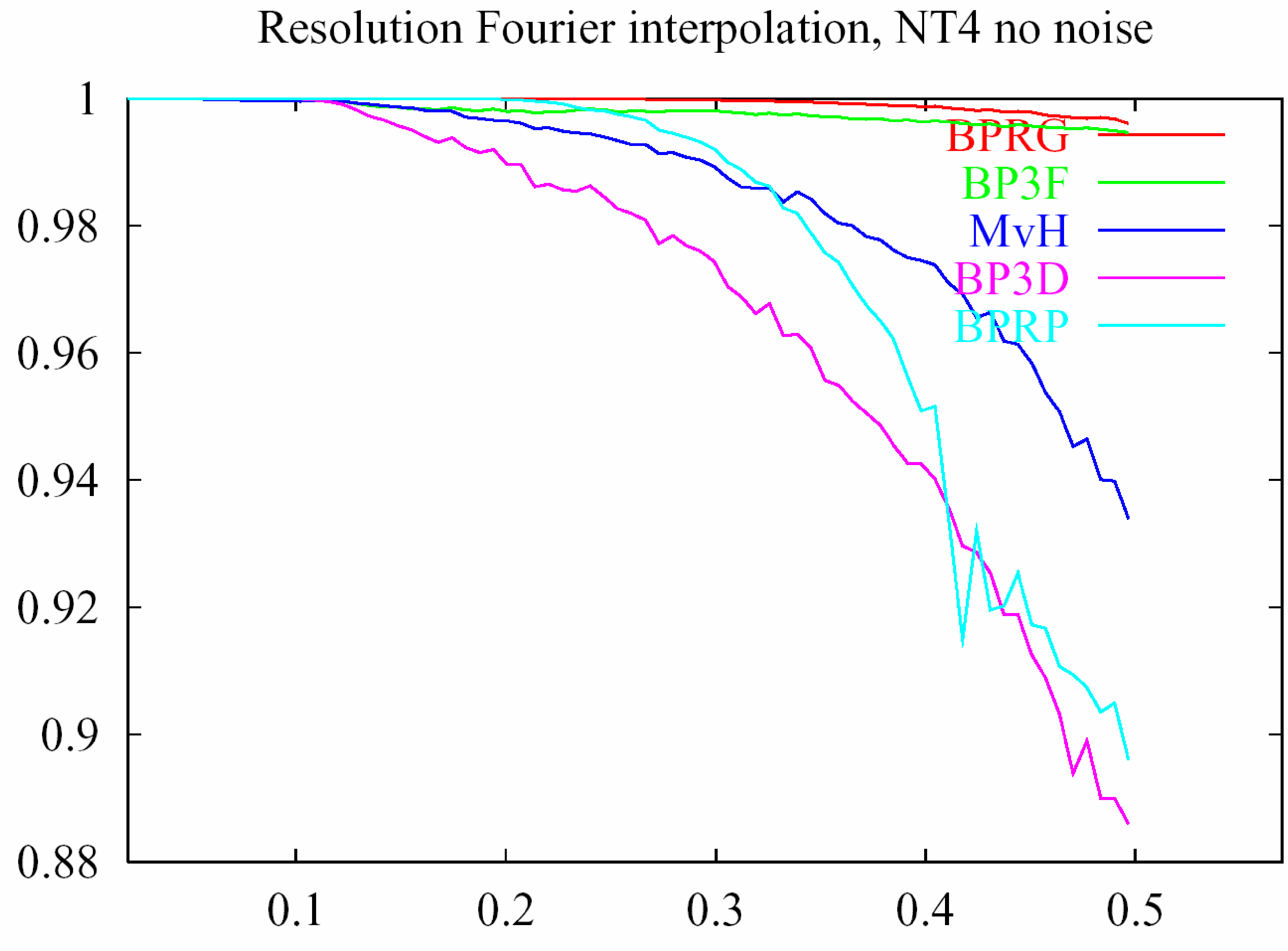
Ranking of the reconstruction methods

- Algebraic methods, particularly Richardson's methods (constant λ) give very good results, but are difficult to use and very slow.

- "General 2D filter" Fourier methods give acceptable results, but the time of calculations increases exponentially with the number of projections.

- 3D Fourier filtration of back-projected data – no working method.

- Fourier interpolation – by far the fastest method, in the presence of noise and in extreme cases results are poor.



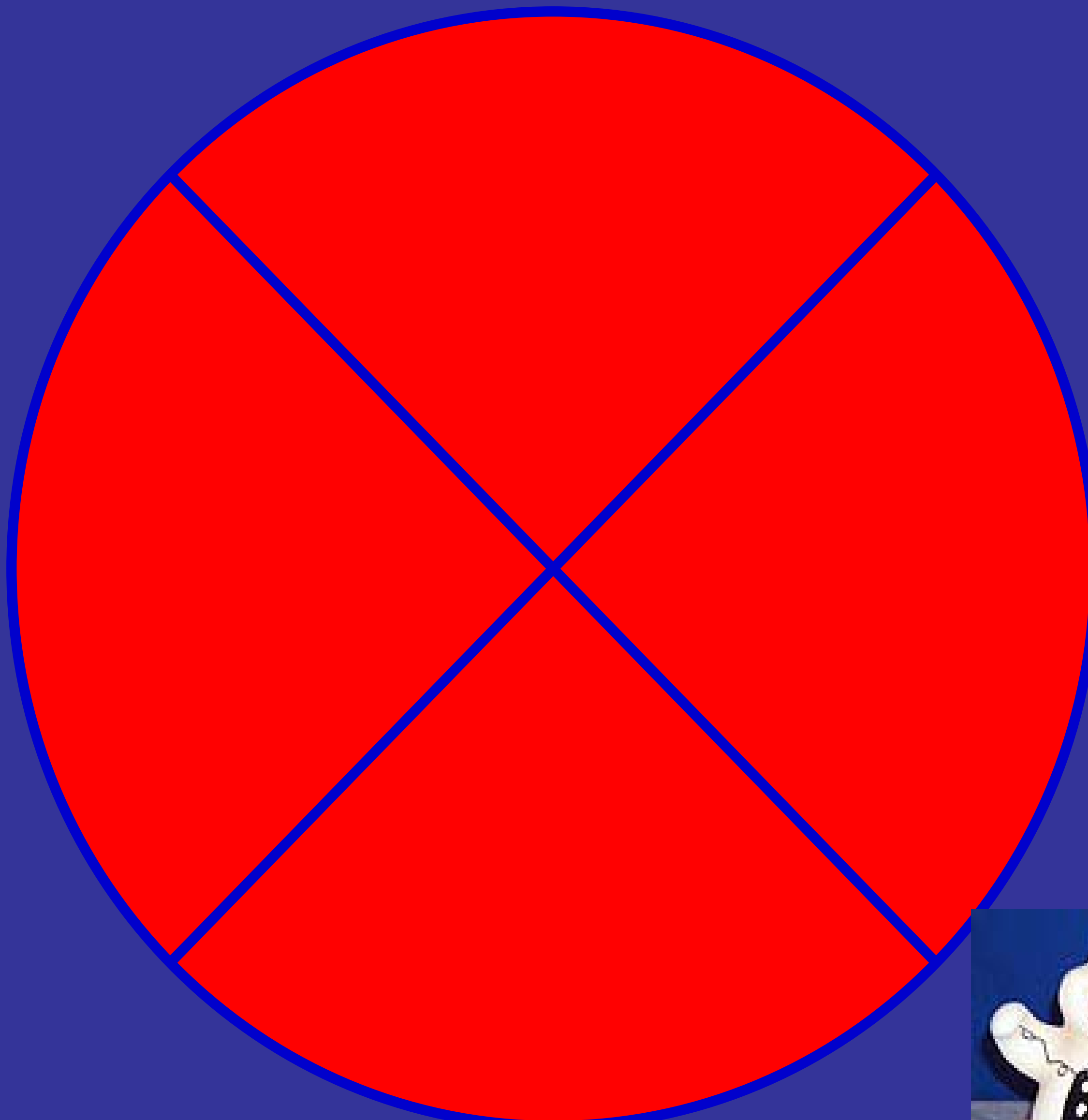
Resolution Fourier interpolation, NT4 no noise
1
0.98
0.96
0.94
0.92
0.9
0.88
0.1
0.2
0.3
0.4
0.5
BPRG
BP3F
MvH
BP3D
BPRP

Summary

- Cryo-EM and single particle analysis rely on the tomographic effect in the electron microscope.

- There is no unique solution to the problem of recovering the 3D structure from the finite set of its 2D projections.

- The quality and speed of 3D reconstruction algorithms differ. Generally, the speed and quality are inversely proportional. Depending on the data set (presence of noise, errors, gaps in angular coverage) some algorithms will perform better than other.




BOO